Exercise 39

Use the Laplace transform to solve the initial boundary-value problem

$$
\begin{aligned}
u_t &= c^2 u_{xx}, \quad 0 < x < a, \ t > 0, \\
u(x,0) &= x + \sin\left(\frac{3\pi x}{a}\right) \quad \text{for } 0 < x < a, \\
u(0,t) &= 0 = u(a,t) \quad \text{for } t > 0.
\end{aligned}
$$

Solution

Solution by the Laplace Transform

The PDE is defined for $t > 0$ and we have an initial condition, so the Laplace transform can be used to solve it. It is defined as

$$\mathcal{L}\{u(x,t)\} = \bar{u}(x,s) = \int_0^t e^{-st} u(x,t)\,dt,$$

which means the derivatives of u with respect to x and t transform as follows.

$$
\begin{aligned}
\mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} &= \frac{d^n \bar{u}}{dx^n} \\
\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= s\bar{u}(x,s) - u(x,0)
\end{aligned}
$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{u_t\} = \mathcal{L}\left\{c^2 u_{xx}\right\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{u_t\} = c^2 \mathcal{L}\{u_{xx}\}$$

Transform the derivatives with the relations above.

$$s\bar{u}(x,s) - u(x,0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

From the initial condition, $u(x,0) = x + \sin(3\pi x/a)$, we have

$$s\bar{u}(x,s) - x - \sin\frac{3\pi x}{a} = c^2 \frac{d^2\bar{u}}{dx^2}.$$

Bring the term with $\bar{u}$ to the other side and divide both sides by c^2.

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = -\frac{1}{c^2}\left(x + \sin\frac{3\pi x}{a}\right) \tag{1}$$

This is an inhomogeneous second order ODE, so the general solution is the sum of the complementary and particular solutions.

$$\bar{u} = \bar{u}_c + \bar{u}_p,$$

$\bar{u}_c$ is the solution to the associated homogeneous ODE,

$$\frac{d^2\bar{u}_c}{dx^2} - \frac{s}{c^2}\bar{u}_c = 0,$$

which can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\bar{u}_c(x,s) = A(s)\cosh\frac{\sqrt{s}}{c}x + B(s)\sinh\frac{\sqrt{s}}{c}x$$

Looking at the inhomogeneous term, we seek a particular solution of the form,
$\bar{u}_p = C_1x + C_2\sin(3\pi x/a)$. Plug this into the ODE in equation (1) to determine C_1 and C_2.

$$-\frac{s}{c^2}C_1x - \left(\frac{9\pi^2}{a^2} + \frac{s}{c^2}\right)C_2\sin\frac{3\pi x}{a} = -\frac{1}{c^2}x - \frac{1}{c^2}\sin\frac{3\pi x}{a}$$

Match the coefficients.

$$\begin{aligned} -\frac{s}{c^2}C_1 &= -\frac{1}{c^2} \quad &\rightarrow \quad C_1 &= \frac{1}{s} \\ -\left(\frac{9\pi^2}{a^2} + \frac{s}{c^2}\right)C_2 &= -\frac{1}{c^2} \quad &\rightarrow \quad C_2 &= \frac{a^2}{9\pi^2c^2 + a^2s} \end{aligned}$$

Hence, we have for the particular solution

$$\bar{u}_p(x,s) = \frac{x}{s} + \frac{a^2}{9\pi^2c^2 + a^2s}\sin\frac{3\pi x}{a}.$$

The general solution to equation (1) is thus

$$\bar{u}(x,s) = A(s)\cosh\frac{\sqrt{s}}{c}x + B(s)\sinh\frac{\sqrt{s}}{c}x + \frac{x}{s} + \frac{a^2}{9\pi^2c^2 + a^2s}\sin\frac{3\pi x}{a}.$$

Our task now is to use the provided boundary conditions at $x = 0$ and $x = a$ to determine $A(s)$ and $B(s)$. Take the Laplace transform of both sides of them.

$$\begin{aligned} u(0,t) = 0 \quad \rightarrow \quad \mathcal{L}\{u(0,t)\} &= \mathcal{L}\{0\} \\ \bar{u}(0,s) &= 0 \quad (2) \\ u(a,t) = 0 \quad \rightarrow \quad \mathcal{L}\{u(a,t)\} &= \mathcal{L}\{0\} \\ \bar{u}(a,s) &= 0 \quad (3) \end{aligned}$$

Plugging $x = 0$ into the general solution and using equation (2), we get

$$\bar{u}(0,s) = A(s) = 0.$$

Plugging $x = a$ into the general solution and using equation (3), we get

$$\bar{u}(a,s) = B(s)\sinh\frac{\sqrt{s}}{c}a + \frac{a}{s} = 0 \quad \rightarrow \quad B(s) = -\frac{a}{s\sinh\frac{\sqrt{s}}{c}a}.$$

Therefore,

$$\bar{u}(x,s) = \frac{x}{s} + \frac{a^2}{9\pi^2c^2 + a^2s}\sin\frac{3\pi x}{a} - \frac{a}{s}\frac{\sinh\frac{\sqrt{s}}{c}x}{\sinh\frac{\sqrt{s}}{c}a}.$$

Now that we have $\bar{u}(x,s)$, we can get $u(x,t)$ by taking the inverse Laplace transform of it.

$$\begin{aligned}
u(x,t) &= \mathcal{L}^{-1}\{\bar{u}(x,s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s} + \frac{a^2}{9\pi^2c^2 + a^2 s}\sin\frac{3\pi x}{a} - \frac{a}{s}\frac{\sinh\frac{\sqrt{s}}{c}x}{\sinh\frac{\sqrt{s}}{c}a}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}x + \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{9\pi^2c^2}{a^2}}\right\}\sin\frac{3\pi x}{a} - a\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{\sinh\frac{\sqrt{s}}{c}x}{\sinh\frac{\sqrt{s}}{c}a}\right\} \\
&= x + e^{-\frac{9\pi^2c^2}{a^2}t}\sin\frac{3\pi x}{a} - a\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}}\right\}
\end{aligned}$$

The inverse Laplace transform of this ratio of hyperbolic sine functions is not located in my table of transforms, so it's necessary to resort to complex variables to solve it. Let

$$\overline{F}(s) = \frac{1}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}}.$$

Since $\sinh z = -i\sin iz$, singularities (that is, where the denominator equals 0) occur where

$$\begin{aligned}
s = 0 \quad &\to \quad s_0 = 0 \\
i\frac{a}{c}\sqrt{s} = n\pi \quad &\to \quad s_n = -\left(\frac{cn\pi}{a}\right)^2, \quad n = 1, 2, \ldots.
\end{aligned}$$

Using Cauchy's residue theorem, we can obtain $F(t)$ by evaluating the residues of $e^{st}\overline{F}(s)$ at these singularities and adding them all together.

$$F(t) = \operatorname*{Res}_{s=s_0}[e^{st}\overline{F}(s)] + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n}[e^{st}\overline{F}(s)]$$

Determine the first term.

$$\operatorname*{Res}_{s=s_0}[e^{st}\overline{F}(s)] = \operatorname*{Res}_{s=0}\frac{e^{st}}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}}$$

Use the Taylor series expansions for each of the functions about $s = 0$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \cdots \\
\sinh x &= x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots
\end{aligned}$$

Substituting these expressions, we get

$$\frac{e^{st}}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}} = \frac{1}{s}(1 + st + \cdots)\frac{\frac{x}{c}\sqrt{s} + \frac{1}{6}\left(\frac{x}{c}\sqrt{s}\right)^3 + \cdots}{\frac{a}{c}\sqrt{s} + \frac{1}{6}\left(\frac{a}{c}\sqrt{s}\right)^3 + \cdots}.$$

The residue at $s = 0$ is the coefficient of the $1/s$ term, so all we need is one term from the long division.

$$\frac{e^{st}}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}} = \frac{1}{s}(1 + st + \cdots)\left(\frac{x}{a} + \cdots\right)$$

Thus,

$$\operatorname*{Res}_{s=s_0}\left[e^{st}\overline{F}(s)\right] = \frac{x}{a}.$$

Now we will determine the second term.

$$\operatorname*{Res}_{s=s_n}\left[e^{st}\overline{F}(s)\right] = \operatorname*{Res}_{s=-\frac{c^2n^2\pi^2}{a^2}} \frac{e^{st}}{s}\frac{\sinh\frac{x}{c}\sqrt{s}}{\sinh\frac{a}{c}\sqrt{s}}$$

Let

$$\begin{aligned} p(s) &= e^{st}\sinh\frac{x}{c}\sqrt{s} \\ q(s) &= s\sinh\frac{a}{c}\sqrt{s}. \end{aligned}$$

Since

$$p(s=s_n) = ie^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a} \neq 0$$

and

$$q(s=s_n) = 0$$

and

$$q'(s=s_n) = \frac{1}{2}(-1)^n in\pi \neq 0,$$

the residue at $s = s_n$ is

$$\begin{aligned} \operatorname*{Res}_{s=s_n}\left[e^{st}\overline{F}(s)\right] &= \frac{p(s_n)}{q'(s_n)} \\ &= \frac{2(-1)^n}{n\pi}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a} \end{aligned}.$$

We now know the inverse Laplace transform of the ratio of hyperbolic sines.

$$F(t) = \operatorname*{Res}_{s=s_0}\left[e^{st}\overline{F}(s)\right] + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n}\left[e^{st}\overline{F}(s)\right] = \frac{x}{a} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a}$$

Plugging this into the formula for $u(x,t)$, we have

$$u(x,t) = x + e^{-\frac{9\pi^2c^2}{a^2}t}\sin\frac{3\pi x}{a} - a\left[\frac{x}{a} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a}\right].$$

Distribute $-a$.

$$u(x,t) = \cancel{x} + e^{-\frac{9\pi^2c^2}{a^2}t}\sin\frac{3\pi x}{a} - \cancel{x} - \sum_{n=1}^{\infty}\frac{2a(-1)^n}{n\pi}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a}$$

Therefore,

$$u(x,t) = e^{-\frac{9\pi^2c^2}{a^2}t}\sin\frac{3\pi x}{a} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a}.$$

Solution by Separation of Variables

In order to verify the solution obtained by the Laplace transform, we'll solve the same exercise with separation of variables. The method of separation of variables can be applied to solve it because the PDE is linear and homogeneous. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE.

$$XT' = c^2 X'' T$$

Bring all terms with t and constants to the left side and all terms with x to the right side.

$$\frac{T'}{c^2 T} = \frac{X''}{X} \tag{4}$$

Now plug the product solution into the boundary conditions. We assume $T(t)$ is not zero.

$$u(0,t) = 0 \quad \rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \tag{5}$$

$$u(a,t) = 0 \quad \rightarrow \quad X(a)T(t) = 0 \quad \rightarrow \quad X(a) = 0 \tag{6}$$

Because the left side of equation (4) is a function of t and the right side is a function of x, the only way both sides can be equal is if they equal a constant. In order to obtain a nontrivial solution for the resulting ODE in x, this constant must be negative.

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

The PDE has thus been reduced to two ODEs, one in t and one in x:

$$T' = -c^2\lambda^2 T \quad \text{and} \quad X'' = -\lambda^2 X.$$

The solution for the ODE in x can be written in terms of sine and cosine.

$$X(x) = C_3 \cos \lambda x + C_4 \sin \lambda x$$

Plug in $x = 0$ and use equation (5) to determine one of the constants.

$$X(0) = C_3 = 0$$

Plug in $x = a$ and use equation (6).

$$X(a) = C_4 \sin \lambda a = 0$$

We assume that $C_4 \neq 0$ because otherwise a trivial solution would result.

$$\sin \lambda a = 0 \quad \rightarrow \quad \lambda a = n\pi \quad \rightarrow \quad \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, \ldots$$

The values of the constant λ_n that satisfy the boundary conditions are called the eigenvalues, and the functions $X_n(x)$ that satisfy the ODE are called the eigenfunctions.

$$X_n(x) = \sin \frac{n\pi x}{a}$$

The solution for the ODE in t can be written in terms of the exponential function.

$$T(t) = C_5 e^{-c^2\lambda^2 t}$$

Plugging in the eigenvalues, we get

$$T_n(t) = C_5 e^{-\frac{c^2 n^2 \pi^2}{a^2} t}.$$

The general solution to the PDE is the sum of all eigenfunctions with their associated eigenvalues (the principle of linear superposition).

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} D_n e^{-\frac{c^2 n^2 \pi^2}{a^2} t} \sin \frac{n\pi x}{a} \end{aligned}$$

To determine this final constant D_n, we make use of the provided initial condition, $u(x,0) = x + \sin(3\pi x/a)$.

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} = x + \sin \frac{3\pi x}{a}$$

We will solve this equation for D_n by taking advantage of the orthogonality of the sine function. Multiply both sides by $\sin(m\pi x/a)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} = \left(x + \sin \frac{3\pi x}{a} \right) \sin \frac{m\pi x}{a}$$

Integrate both sides with respect to x over the domain it is defined—from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dx = \int_0^a \left(x + \sin \frac{3\pi x}{a} \right) \sin \frac{m\pi x}{a}\, dx$$

Bring the integral inside the sum on the left.

$$\sum_{n=1}^{\infty} D_n \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dx = \int_0^a \left(x + \sin \frac{3\pi x}{a} \right) \sin \frac{m\pi x}{a}\, dx$$

Unless $n = m$, the integral of the product of sine functions is 0 because of orthogonality, so every term in the series vanishes except for the one where $n = m$.

$$D_n \int_0^a \sin^2 \frac{n\pi x}{a}\, dx = \int_0^a \left(x + \sin \frac{3\pi x}{a} \right) \sin \frac{n\pi x}{a}\, dx$$

Evaluate the integral on the left side and split up the integral on the right.

$$D_n \frac{a}{2} = \int_0^a x \sin \frac{n\pi x}{a}\, dx + \int_0^a \sin \frac{3\pi x}{a} \sin \frac{n\pi x}{a}\, dx$$

Evaluate the first integral on the right side.

$$D_n \frac{a}{2} = \frac{(-1)^{n+1}}{n} \frac{a^2}{\pi} + \int_0^a \sin \frac{3\pi x}{a} \sin \frac{n\pi x}{a}\, dx$$

Multiply both sides by $2/a$ to solve for D_n.

$$D_n = \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} + \frac{2}{a}\int_0^a \sin\frac{3\pi x}{a}\sin\frac{n\pi x}{a}\,dx$$

Unless $n = 3$, the last integral vanishes. When $n = 3$, we have

$$\begin{aligned} D_3 &= \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} + \frac{2}{a}\int_0^a \sin^2\frac{3\pi x}{a}\,dx \\ &= \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} + \frac{2}{a}\cdot\frac{a}{2} \\ &= \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} + 1 \end{aligned}.$$

Thus,

$$D_n = \begin{cases} \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} & n \neq 3 \\ \frac{(-1)^{n+1}}{n}\frac{2a}{\pi} + 1 & n = 3 \end{cases}.$$

Therefore,

$$u(x,t) = e^{-\frac{9\pi^2c^2}{a^2}t}\sin\frac{3\pi x}{a} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{c^2n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a},$$

which is the same answer we obtained using the Laplace transform.